

Functional Schrödinger and BRST Quantization of (1+1)–Dimensional Gravity *

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Abstract

We discuss the quantization of pure string–inspired dilaton–gravity in (1+1)–dimensions, and of the same theory coupled to scalar matter. We perform the quantization using the functional Schrödinger and BRST formalisms. We find, both for pure gravity and the matter–coupled theory, that the two quantization procedures give inequivalent “physical” results.

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1 Introduction

In four space–time dimensions, gravitational theories are non–renormalizable, which makes their quantization problematic using conventional field–theoretical techniques. In addition, there are conceptual difficulties that are peculiar to diffeomorphism–invariant quantum theories like gravity. These include the question of how to introduce time into the theory, and the interpretational issues that arise from the unfamiliar role of the Hamiltonian as a constraint. Recently, much work has been done on gravitational theories in $(1 + 1)$ –dimensions, in which the computational difficulties that one faces in $(3 + 1)$ –dimensions are absent, because there are no propagating gravitons in the lower dimensional theory, and the question of renormalizability does not arise. The conceptual issues remain, however, and the $(1 + 1)$ –dimensional models are useful for investigating them.

In two space–time dimensions the Einstein tensor vanishes identically, so $(1 + 1)$ –dimensional gravity models cannot be based on the Einstein–Hilbert action. A variety of models have been proposed, the most popular of which is the string–inspired CGHS model [1, 2], which belongs to the class of scalar–tensor theories introduced over a decade ago [3]. Our paper is concerned with a theory, related by a conformal transformation to the CGHS model, and described by the action

$$S = \int dt \int d\sigma \sqrt{-g}(\eta R - 2\lambda). \quad (1.1)$$

This theory has been quantized, using both the metric–based action (1.1) [4] and an equivalent gauge–theoretical action, invariant under the extended Poincaré group [5, 6]. The quantization consists of solving the Dirac constraints in a functional Schrödinger formulation.

Recent work on the string–inspired model [7] has shown that it is similar to the

theory of a bosonic string propagating on a $(1 + 1)$ -dimensional background space-time. It is a familiar result from string theory that the bosonic string cannot be straightforwardly quantized on a two-dimensional target space, because the Virasoro anomaly gives a center to the algebra of constraints. In light of this, it is surprising that solutions to the quantum constraint algebra derived from (1.1) can be constructed [4, 6], since their existence implies that the center vanishes.

The resolution of the apparent contradiction [7] relies on the fact that the anomaly is present or absent depending on the vacuum that is used to “normal” order the constraints: the conventional vacuum of string theory is not appropriate for the functional Schrödinger representation that was used in [4, 6, 7].

However, string-like theories in $(1 + 1)$ -dimensions have been constructed, using the BRST approach [8, 9, 10, 11]. They are made consistent through the addition of background charges and ghost fields, and the resulting spectrum contains more states than the ones found in the gravitational theory: there is a continuous component, consisting of two families of states labeled by the zero-mode momenta of the fields; in addition, there is an infinite tower of “discrete states” that appear at special values of the zero-mode momenta.

We shall show that the continuous component of the BRST spectrum finds its analog in the functional Schrödinger quantization when we solve the theory (1.1) on the cylindrical space-time $\mathbf{R}^1 \times S^1$. The tower of discrete states, on the other hand, arises only when background charges are present. Since there is no center in the functional Schrödinger ordering, there is no need to introduce background charges, and we find only two additional discrete states with vanishing zero-mode momentum.

We shall present a similar comparison for the matter-coupled theory. We use the results of [7] to show that the action (1.1), treated in the functional Schrödinger

formalism, and minimally coupled to scalar matter, is equivalent to an action studied in [12]. The theory cannot be quantized without modification, due to an obstruction in the matter field contribution to the constraint algebra. A consistent quantum theory is constructed by adding terms to the constraints that canonically yield a center in the Poisson bracket algebra of constraints, which cancels the center in the quantum theory that arises from ordering the matter field contribution to the constraint commutator algebra [7, 12]. Furthermore, we consider the alternative BRST approach where the total non-vanishing center arises from the gravitational and matter fields, as well as from background charges, and is canceled by the ghost contribution.

In the remainder of this Section, we review the transformations on (1.1) that bring out the analogy to $(1+1)$ -dimensional string theory. Also we discuss the origin of the anomalies in the algebra of constraints and show that they are absent in the Schrödinger representation, by recording explicitly the states on \mathbf{R}^2 that solve the constraints.

To put (1.1) in canonical form, we parameterize the metric tensor $g_{\mu\nu}$ as in [4],

$$g_{\mu\nu} = e^{2\rho} \begin{pmatrix} u^2 - v^2 & v \\ v & -1 \end{pmatrix}. \quad (1.2)$$

The variables u and v enter the action without time derivatives, and act as Lagrange multipliers, enforcing the diffeomorphism constraints; they are related to the shift and lapse functions. The dynamical canonical fields are ρ , the logarithm of the conformal factor, and η , the dilaton. In first-order form, Eq. (1.1) becomes

$$S = \int dt \int d\sigma (\Pi_\rho \dot{\rho} + \Pi_\eta \dot{\eta} - \mathcal{H}), \quad (1.3)$$

where an over-dot indicates derivative with respect to time t . The Hamiltonian density \mathcal{H} is a sum of constraints, expressed in terms of canonical coordinates ρ , η , their momenta Π_ρ , Π_η , and the Lagrange multipliers u , v ;

$$\mathcal{H} = u\mathcal{E} + v\mathcal{P}, \quad (1.4)$$

where

$$\mathcal{E} = -2(\eta'' - \rho'\eta') + \frac{1}{2}\Pi_\rho\Pi_\eta + 2\lambda e^{2\rho} \quad (1.5)$$

$$\mathcal{P} = -\Pi_\rho\rho' + \Pi'_\rho - \Pi_\eta\eta' . \quad (1.6)$$

We use a prime to indicate differentiation with respect to the spatial coordinate σ .

The action (1.3) is equivalent to that describing two free scalar and Hermitian fields r^a , $\{a = 0, 1\}$, with indefinite metric $\eta_{ab} = \text{diag}(1, -1)$ [7]. We demonstrate this by making a canonical redefinition from ρ , η , Π_ρ and Π_η to r^a and π_a (at fixed time t , whose label is suppressed),[†]

$$\pi_0 - \lambda r^{1'} = 2\lambda e^\rho \sinh \Sigma \quad (1.7a)$$

$$\pi_1 + \lambda r^{0'} = -2e^\rho \cosh \Sigma \quad (1.7b)$$

$$\lambda r^0 = -\frac{1}{2}e^{-\rho}(2\eta' \cosh \Sigma - \Pi_\rho \sinh \Sigma) \quad (1.7c)$$

$$\lambda r^1 = \frac{1}{2}e^{-\rho}(\Pi_\rho \cosh \Sigma - 2\eta' \sinh \Sigma) , \quad (1.7d)$$

where

$$\Sigma(\sigma) = \frac{1}{2} \int_{-\infty}^{\sigma} d\tilde{\sigma} \Pi_\eta(\tilde{\sigma}). \quad (1.8)$$

In terms of the new fields π_a , r^a , Eq. (1.3) becomes

$$S = \int dt \int d\sigma (\pi_a \dot{r}^a - \mathcal{H}) , \quad (1.9)$$

with \mathcal{H} as in (1.4), but now

$$\begin{aligned} \mathcal{E} &= -\frac{1}{2} \left(\frac{1}{\lambda} \pi^a \pi_a + \lambda r^{a'} r_a' \right) \\ &= -\frac{1}{2} \left(\frac{1}{\lambda} (\pi_0)^2 + \lambda (r^{0'})^2 \right) + \frac{1}{2} \left(\frac{1}{\lambda} (\pi_1)^2 + \lambda (r^{1'})^2 \right) \\ &= -\mathcal{E}^0 + \mathcal{E}^1 , \end{aligned} \quad (1.10)$$

[†]The overlap matrix elements between the two sets of fields are computed in [13].

$$\begin{aligned}
\mathcal{P} &= -\pi_a r^{a'} = -\pi_0 r^{0'} - \pi_1 r^{1'} \\
&= \mathcal{P}^0 + \mathcal{P}^1 .
\end{aligned} \tag{1.11}$$

(Note the sign variation in \mathcal{E} ; this leads to the variety of vacua, whose properties we study in this paper.)

The theory defined by (1.9) has been quantized using the Dirac procedure [7], in a functional Schrödinger representation. When we work with this formalism, by letting the momenta π_a act by functional differentiation,

$$\pi_a \rightarrow \frac{1}{i} \frac{\delta}{\delta r^a} , \tag{1.12}$$

on wave functionals $\Psi(r^a)$ that depend on r^a , which act by multiplication, quantization consists of solving the constraint conditions

$$\mathcal{E}\Psi = \mathcal{P}\Psi = 0 . \tag{1.13}$$

There are two solutions Ψ_{\pm} ,

$$\Psi_{\pm} = \exp \left[\pm i \frac{\lambda}{2} \int_{-\infty}^{\infty} d\sigma \, r^a \epsilon_{ab} r^{b'} \right] . \tag{1.14}$$

Since a solution to the constraints exists explicitly, it follows that the constraint algebra has no obstruction, and satisfies the naive commutation relations, without center,

$$i[\mathcal{E}(\sigma), \mathcal{E}(\tilde{\sigma})] = i[\mathcal{P}(\sigma), \mathcal{P}(\tilde{\sigma})] = \left(\mathcal{P}(\sigma) + \mathcal{P}(\tilde{\sigma}) \right) \delta'(\sigma - \tilde{\sigma}) , \tag{1.15a}$$

$$i[\mathcal{E}(\sigma), \mathcal{P}(\tilde{\sigma})] = \left(\mathcal{E}(\sigma) + \mathcal{E}(\tilde{\sigma}) \right) \delta'(\sigma - \tilde{\sigma}) . \tag{1.15b}$$

These commutators are obtained by applying the canonical commutation relations

$$i[\pi_a(\sigma), r^b(\tilde{\sigma})] = \delta_a^b \delta(\sigma - \tilde{\sigma}) \tag{1.16}$$

and ignoring issues of ordering.

A more careful analysis, which takes into account operator product singularities, exposes the possibility of a center in (1.15b)

$$i[\mathcal{E}(\sigma), \mathcal{P}(\tilde{\sigma})] = \left(\mathcal{E}(\sigma) + \mathcal{E}(\tilde{\sigma}) \right) \delta'(\sigma - \tilde{\sigma}) - \frac{c}{12\pi} \delta'''(\sigma - \tilde{\sigma}). \quad (1.17)$$

We may now understand why the above analysis, leading to the solution (1.14), is not obstructed by a center. The indefinite signs of the energy constraint (1.10) imply that $[\mathcal{E}, \mathcal{P}] = -[\mathcal{E}^0, \mathcal{P}^0] + [\mathcal{E}^1, \mathcal{P}^1]$, so that if identical centers arise for both “0” and “1” fields, they cancel.

We can write (1.9) in second order form as the action of two free scalar fields. The action \overline{S} ,

$$\overline{S} = -\frac{\lambda}{2} \int dt \int d\sigma \sqrt{-g} g^{\mu\nu} \partial_\mu r^a \partial_\nu r^b \eta_{ab} \quad (1.18)$$

equals S of (1.9) when the metric is parameterized as in Eq. (1.2), and \overline{S} is put into canonical form. The Hamiltonian density is then identical to that of Eq. (1.4), with the constraints given by Eqs. (1.10,1.11).

When the gravitational theory is expressed as in (1.18) it resembles a bosonic string theory. Indeed if the model is defined on the cylinder $\mathbf{R}^1 \times S^1$, rather than on \mathbf{R}^2 , we have a closed bosonic string in $\{t, \sigma\}$ parameter space, propagating on a flat two-dimensional target space r^a , with Minkowski metric tensor η_{ab} . (It is interesting to note that the cosmological constant λ has become the string tension.) As discussed in the introductory paragraphs, the string theory cannot be quantized, due to the anomaly in the constraint algebra (1.17). The anomaly in string theory is insensitive to the signature of the target space, and our indefinite metric would play no role. This is not in contradiction with the fact that we found the solutions (1.14), above. As shown in [7], the anomaly is present when the constraints are ordered with respect to the conventional string theoretic vacuum, and absent when they are ordered with respect

to the functional Schrödinger vacuum.

In Section 2 we reformulate the gravity theory on $\mathbf{R}^1 \times S^1$ and give a mode decomposition, so that the formalism coincides with that of string theory. We review the normal ordering prescription (choice of vacuum) used in string theory resulting in a center, and the one used in the Schrödinger representation, which does not produce a center.

In Section 3 the $\mathbf{R}^1 \times S^1$ theory is quantized with Schrödinger–representation normal ordering; states analogous to (1.14) are constructed, and further states are found, which exist only on the cylindrical geometry. These give a representation for the Virasoro algebra without center, which is absent with the chosen ordering.

Section 4 is devoted to the string–type quantization. A center in the Virasoro algebra exists; the operators are modified by the addition of background charges, thereby increasing the center; ghost fields are added with negative center so that the resulting total center vanishes. BRST invariant states, which are deemed “physical”, are constructed. The resulting spectrum is richer than what is found in Section 3 [8, 9, 10, 11]; however, the additional “discrete” states arise at imaginary values of the zero–mode momentum, so the fields in these theories transform differently under Hermitian conjugation than those in Section 3. It is the combination of the different Hermiticity properties of the fields and the presence of background charges that is responsible for the infinite tower of discrete states in the spectrum. When we demand that the fields be Hermitian, we find that there are no discrete states: only the continuous portion of the BRST spectrum is present. Consequently, we find fewer states with this approach than in the functional Schrödinger quantization.

In Section 5, we consider the dilaton–gravity action (1.1) coupled to a massless scalar field. This theory possesses a center for either choice of ordering, arising from

the matter degrees of freedom [7]. It was shown in [7] that in the functional Schrödinger formulation the matter-coupled action is equivalent to one studied in [12], in which the theory was quantized after modifying it to remove the center. In subSection 5.1 we reproduce the argument of [12], and present the spectrum. In subSection 5.2 we quantize the theory using the BRST approach, adopting a string-like ordering for the fields. We introduce background charges to increase the total, positive center, which cancels against that of the ghost fields. We then follow the development in [9, 11, 14] to compute the spectrum, again omitting states with imaginary momentum. We find that the resulting set of states is larger than that presented in subSection 5.1, because of an additional, unconstrained, zero-mode degree of freedom present in the BRST spectrum.

In Section 6 we summarize and comment upon our results.

2 Mode Analysis and Vacua

With the action (1.18) we work in conformal gauge, which we fix by setting $u = 1$, $v = 0$ in the parameterized metric tensor $g_{\mu\nu}$, given by (1.2). The action then describes two free scalar fields r^a , one entering with negative kinetic term, the other with positive kinetic term (it is this alternation of sign that leads to the variety of vacuum choices),

$$\overline{S} = \frac{\lambda}{2} \int dt \int d\sigma \left\{ -\left((\dot{r}^0)^2 - (r^{0\prime})^2\right) + \left((\dot{r}^1)^2 - (r^{1\prime})^2\right) \right\}. \quad (2.1)$$

The equations of motion

$$\frac{\delta \overline{S}}{\delta r^a} = 0 \Rightarrow \ddot{r}^a - r^{a''} = 0 \quad (2.2)$$

are solved by arbitrary functions of $t \pm \sigma$, and we apply spatial periodic boundary conditions: the spatial interval is taken to be of length 2π , and henceforth we set $\lambda = 1/4\pi$ for simplicity. We expand the solution in terms of mode operators, consistent

with the periodicity requirement,

$$r^a(t, \sigma) = \hat{x}^a + 2t\hat{p}^a + i \sum_{n \neq 0} \frac{1}{n} \left[\alpha_n^a e^{-in(t-\sigma)} + \bar{\alpha}_n^a e^{-in(t+\sigma)} \right] . \quad (2.3)$$

The commutation relations (1.16) with $\pi_a = -\frac{1}{4\pi} \dot{r}_a$, imply the algebra

$$[\hat{p}^a, \hat{x}^b] = i\eta^{ab} \quad (2.4)$$

$$[\alpha_m^a, \alpha_n^b] = [\bar{\alpha}_m^a, \bar{\alpha}_n^b] = -m\eta^{ab}\delta_{m+n,0} . \quad (2.5)$$

The formal expressions for the constraints that would be obtained by varying with respect to the multipliers u and v , which are now fixed, coincide with (1.10), (1.11).

We express them in terms of mode operators

$$L_m \equiv \frac{1}{2} \int_0^{2\pi} d\sigma e^{-im\sigma} (\mathcal{E} + \mathcal{P}) \Big|_{t=0} = -\eta_{ab} L_m^{ab} \quad (2.6a)$$

$$\bar{L}_m \equiv \frac{1}{2} \int_0^{2\pi} d\sigma e^{im\sigma} (\mathcal{E} - \mathcal{P}) \Big|_{t=0} = -\eta_{ab} \bar{L}_m^{ab} , \quad (2.6b)$$

$$L_m^{ab} = \frac{1}{2} \sum_n : \alpha_{m+n}^a \alpha_{-n}^b : , \quad \bar{L}_m^{ab} = \frac{1}{2} \sum_n : \bar{\alpha}_{m+n}^a \bar{\alpha}_{-n}^b : . \quad (2.7)$$

The colons denote an as yet unspecified “normal” ordering rule. Also $\alpha_0^a = \bar{\alpha}_0^a \equiv \hat{p}^a$. The structures associated with the “barred” expressions are identical with the un-“barred” ones, and the two commute with each other. But they do not act independently of one another, since the zero mode oscillators α_0^a and $\bar{\alpha}_0^a$ are identified.

The operators L_m obey the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} , \quad (2.8)$$

and similarly for \bar{L}_m . The central element c is determined by the normal-ordering prescription. From this expression we see that the constraint conditions $L_m|\psi\rangle = \bar{L}_m|\psi\rangle = 0$ are consistent only if $c = 0$.

We determine the center in the algebra (2.8) by specifying a vacuum state. The convention adopted in string theory is that the vacuum $|p^a\rangle_{\text{ST}}$ is annihilated by the positive-frequency mode operators,

$$\alpha_n^a |p^a\rangle_{\text{ST}} = \bar{\alpha}_n^a |p^a\rangle_{\text{ST}} = 0, \quad n > 0, \quad (2.9a)$$

and is an eigenstate of the zero-mode momentum operator \hat{p}^a ,

$$\hat{p}^a |p^a\rangle_{\text{ST}} = p^a |p^a\rangle_{\text{ST}}. \quad (2.9b)$$

With this choice, the value of the center is independent of the signature of the target-space metric η_{ab} . The total center is therefore twice that for a single scalar field, $c = 2$, and as a consequence the constraint conditions cannot be solved. Furthermore, the string vacuum $|p^a\rangle_{\text{ST}}$ also gives rise to states of negative norm [7]. This property makes it unsuitable for the functional Schrödinger approach, since in that formulation the norm of a state is given by a manifestly positive functional integral. Moreover, since $\alpha_{|n|}^0$ corresponds to $\int_0^{2\pi} d\sigma e^{-i|n|\sigma} \left(\pi_0 + \frac{|n|}{4\pi} r^0 \right)$ (at $t = 0$), a state annihilated by $\alpha_{|n|}^0$ satisfies in the Schrödinger representation the equation

$$i \int_0^{2\pi} d\sigma e^{-i|n|\sigma} \left(-\frac{\delta}{\delta r^0} + \frac{|n|}{4\pi} r^0 \right) \Psi_{\text{ST}}^{\text{vac}} = 0, \quad n \neq 0. \quad (2.10)$$

This is solved by a quadratic exponential, which grows in function space, and does not describe a localized, normalizable state. ($\Psi_{\text{ST}}^{\text{vac}}$ is the vacuum wave functional corresponding to the abstract state $|p^a\rangle_{\text{ST}}$.)

In the functional Schrödinger formalism, an alternative vacuum, $|p^a\rangle_{\text{FS}}$, is adopted for the “0” variables with negatively-signed kinetic term. This vacuum satisfies

$$\alpha_{-n}^0 |p^a\rangle_{\text{FS}} = \alpha_n^1 |p^a\rangle_{\text{FS}} = \bar{\alpha}_{-n}^0 |p^a\rangle_{\text{FS}} = \bar{\alpha}_n^1 |p^a\rangle_{\text{FS}} = 0, \quad n > 0, \quad (2.11a)$$

$$\hat{p}^a |p^a\rangle_{\text{FS}} = p^a |p^a\rangle_{\text{FS}}. \quad (2.11b)$$

For the “0” excitations, the negative-frequency mode is taken to annihilate the vacuum; since it contributes to the energy constraint with a negative sign, in a sense it is equivalent to the positive-frequency mode of the “1” excitations. Moreover, the annihilation requirement, in contrast to (2.10), now demands

$$i \int_0^{2\pi} d\sigma e^{i|n|\sigma} \left(\frac{\delta}{\delta r^0} + \frac{|n|}{4\pi} r^0 \right) \Psi_{\text{FS}}^{\text{vac}} = 0 \quad , \quad n \neq 0 \quad , \quad (2.12)$$

which provides Gaussian, localized solutions. ($\Psi_{\text{FS}}^{\text{vac}}$ is the vacuum wave functional corresponding to the abstract state $|p^a\rangle_{\text{FS}}$.) The solution for the vacuum wave functional, satisfying (2.11a) and the zero-mode condition (2.11b), which becomes $(p_a + \int_0^{2\pi} d\sigma \pi_a) \Psi_{\text{FS}}^{\text{vac}} = 0$, is

$$\Psi_{\text{FS}}^{\text{vac}} = e^{-\frac{i}{2\pi} p_a \int_0^{2\pi} d\sigma r^a} \left[-\frac{1}{2} \int_0^{2\pi} d\sigma \int_0^{2\pi} d\tilde{\sigma} (r^0 \omega r^0 + r^1 \omega r^1) \right] \quad , \quad (2.13)$$

where

$$\omega(\sigma, \tilde{\sigma}) = \frac{1}{8\pi^2} \sum_n |n| e^{in(\sigma - \tilde{\sigma})} = -\frac{1}{16\pi^2} \text{P} \frac{1}{\sin^2 \frac{1}{2}(\sigma - \tilde{\sigma})} \quad . \quad (2.14)$$

With this choice, there are no negative-normed states; moreover, the center is zero, because the one coming from the “0” oscillator cancels against that of the “1” oscillator. Evidently the spatial integral of the \mathcal{E} operator is not positive definite, because it vanishes on physical states by virtue of a cancelation, but this is not a defect in our gravity theory, since $\int d\sigma \mathcal{E}$ is not the energy of a “physical” state.

3 Schrödinger–Dirac Quantization on the Cylinder

For the cylindrical geometry, we now construct states that are annihilated by the Virasoro generators, which are ordered without a center, as in (2.11a,b). The constraints are most readily solved by writing the Virasoro operators as a product of factors. We shall display the calculations only for the unbarred operators L_m ; similar expressions can be constructed for the barred ones.

We define light-cone combinations of the mode operators [‡]

$$\alpha_m^\pm = \frac{1}{\sqrt{2}}(\alpha_m^0 \pm \alpha_m^1) \quad (3.1a)$$

$$\hat{p}^\pm = \frac{1}{\sqrt{2}}(\hat{p}^0 \pm \hat{p}^1) \quad (3.1b)$$

$$\hat{x}^\pm = \frac{1}{\sqrt{2}}(\hat{x}^0 \pm \hat{x}^1) , \quad (3.1c)$$

and find that they satisfy

$$[\alpha_m^\pm, \alpha_n^\pm] = 0 , \quad [\alpha_m^+, \alpha_n^-] = -m \delta_{m,-n} , \quad (3.2a)$$

$$[\hat{p}^\pm, \hat{x}^\pm] = 0 , \quad [\hat{p}^\pm, \hat{x}^\mp] = i . \quad (3.2b)$$

In terms of these, the Virasoro operators appear in their factorized form,

$$L_m = - \sum_n : \alpha_{m+n}^+ \alpha_{-n}^- : . \quad (3.3)$$

From (3.2a) and (3.3), we see that the normal-ordering symbol only affects L_0 , so the solutions $|\pm\rangle$ to the set of equations

$$\alpha_m^\pm |\pm\rangle = 0 , \quad m \text{ a nonzero integer} \quad (3.4a)$$

$$\hat{p}^\pm |\pm\rangle = 0 , \quad (3.4b)$$

are annihilated by all of the Virasoro operators, with the possible exception of L_0 . That they are annihilated by L_0 , as well, can be verified directly or by applying the Virasoro algebra (2.8), with $c = 0$.

The solutions to Eqs. (3.4a,b) form two families (\pm) , labeled by the zero-mode momentum,

$$|\pm\rangle = \exp\left(\mp \sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n}^1 \alpha_n^0\right) |p^\mp\rangle_{\text{FS}} \equiv e^{\mp\Omega} |p^\mp\rangle_{\text{FS}} , \quad (3.5)$$

[‡]Our notation differs from that of [7], where the “ \pm ” superscript refers to oscillators with positive and negative kinetic term, while here we use it to denote light-cone combinations.

where $\widehat{p}^\pm |p^\pm\rangle_{\text{FS}} = p^\pm |p^\pm\rangle_{\text{FS}}$, and $\widehat{p}^\mp |p^\pm\rangle_{\text{FS}} = 0$. This is annihilated by the L_m 's; since a similar expression holds for the barred variables, the solution to the full set of constraints, L_m and \overline{L}_m , is

$$e^{\mp\Omega} e^{\mp\overline{\Omega}} |p^\mp\rangle_{\text{FS}} = \exp\left[\mp \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^1 \alpha_n^0 + \overline{\alpha}_{-n}^1 \overline{\alpha}_n^0)\right] |p^\mp\rangle_{\text{FS}}. \quad (3.6)$$

The two additional states $\Psi_\pm(r^a) = \langle r^a | \Psi_\pm \rangle$, corresponding to (1.14), appear in terms of oscillators as [7]

$$|\Psi_\pm\rangle = \exp\left[\mp \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^1 \alpha_n^0 - \overline{\alpha}_{-n}^1 \overline{\alpha}_n^0)\right] |0\rangle_{\text{FS}}. \quad (3.7)$$

There is a relative difference of a minus sign between the terms depending on the barred and unbarred oscillators in (3.6) and (3.7). The negative sign preceding the barred oscillators in (3.7) can only appear when $p^a = 0$. The spectrum therefore contains a continuous component (3.6) and a discrete component, consisting of the two states (3.7).

The wave functionals that correspond to the solutions (3.6) can be obtained by writing the conditions (3.4a,b) in terms of the fields r^a ,

$$\langle r^a | e^{\mp\Omega} e^{\mp\overline{\Omega}} |p^\mp\rangle_{\text{FS}} = e^{-\frac{ip^\mp}{2\pi} \int_0^{2\pi} d\sigma r^\pm} f^\mp(r^\pm), \quad (3.8)$$

where f^\mp is independent of r^\mp and is localized (by a δ -like functional) in r^\pm .

4 BRST Quantization with Background Charges

The constraint algebra (2.8) with $c = 2$ shows that a consistent quantization condition for states built on the vacuum $|p^a\rangle_{\text{ST}}$, defined by (2.9a,b), does not exist because the center is nonvanishing. Nevertheless, the corresponding normal-ordering prescription has been used to quantize theories that are closely related to (2.1) using BRST quantization [8, 9, 10, 11]. The center is removed by adding background charges and ghost

fields. The resulting spectrum of “physical” states (the BRST cohomology) is larger than the one that we found in the functional Schrödinger formulation: there is a continuous component, corresponding to the states (3.6); there is also a discrete component, consisting of an infinite tower of states at specific values of the zero-mode momentum p^a . In a sense, these discrete states are analogous to the two states (3.7); however, in BRST quantization, the zero-mode momenta that label the discrete states are imaginary when the background charges are real, so the fields that enter the theory do not transform in a simple way under Hermitian conjugation, in contrast to the Hermitian fields in (2.1). Using the results of [9], and demanding that the fields be Hermitian, we find that only the continuous component of the BRST spectrum is present, and there are no discrete states. More precisely, there are states in the BRST cohomology that exist for discrete imaginary values of the zero-mode momenta, but these states are unphysical in our context and we omit them from the spectrum.[§] We now give the details of the above argument,

We remove the obstruction in the Virasoro algebra by adding background charges Q^a to the operators (2.7). We define new operators L'_m and \bar{L}'_m (we record the formulae only for the unbarred operators; similar expressions hold for the barred ones)

$$L'_m = -\frac{1}{2}\eta_{ab} \sum_{n \neq -m, 0} : \alpha_{m+n}^a \alpha_{-n}^b : - \hat{p}_a \alpha_m^a - im Q_a \alpha_m^a, \quad m \neq 0, \quad (4.1a)$$

$$L'_0 = -\frac{1}{2}\eta_{ab} \sum_{n \neq 0} : \alpha_n^a \alpha_{-n}^b : - \frac{1}{2} \hat{p}_a \hat{p}^a - \frac{1}{2} Q_a Q^a. \quad (4.1b)$$

We note that this is not the form in which the constraints are presented in [9]. To make contact with that paper, we define new momentum operators P^a and background

[§] The states with imaginary momentum appear to play a role [15, 16] when the wave functional of the universe is evolved using the Euclidean action (as in the Hartle–Hawking construction [17]). Also, for closely related reasons, such states arise when one attempts to cure the non-normalizability of the string-theoretic vacuum by functionally integrating over imaginary field configurations [16] (however, see the discussion in [14].)

charges \tilde{Q}^a by $P^a \equiv \hat{p}^a - iQ^a$ and $\tilde{Q}^a \equiv -iQ^a$, in terms of which (4.1a,b) become

$$L'_m = -\frac{1}{2}\eta_{ab} \sum_{n \neq -m, 0} : \alpha_{m+n}^a \alpha_{-n}^b : - P_a \alpha_m^a + (m+1) \tilde{Q}_a \alpha_m^a, \quad m \neq 0, \quad (4.2a)$$

$$L'_0 = -\frac{1}{2}\eta_{ab} \sum_{n \neq 0} : \alpha_n^a \alpha_{-n}^b : - \frac{1}{2} P_a P^a + P_a \tilde{Q}^a. \quad (4.2b)$$

The contribution of the “1” field to L'_m is then the same expression that is found in [9] for the “matter” field. The contribution from the “0” field, which enters with a negative sign, can be identified with that of the “Liouville” field of [9], after making the change $\alpha_m^0 \rightarrow i\alpha_m^0$: despite the emphasis that we are placing on the role of Hermiticity in determining the spectrum, this alteration does not affect the conclusions of this Section.

In the following we take Q^a in (4.1a,b) to be real. From (4.1a) we see that L'_m is Hermitian if the oscillators satisfy $\alpha_m^{a\dagger} = \alpha_{-m}^a$, which also is the condition appropriate for a Hermitian field. The spectrum that is found in [8, 9, 10, 11] contains states at imaginary eigenvalues of the zero-mode momentum \hat{p}^a . We shall therefore consider two cases: \hat{p}^a anti-Hermitian, and \hat{p}^a Hermitian [the latter is the appropriate one for the theory described by (2.1)]. The Hermiticity conditions that we apply to the Virasoro operators for anti-Hermitian \hat{p}^a are [11] $L_m^\dagger(\hat{p}^a) = L_{-m}(-\hat{p}^a)$, while for Hermitian zero-mode momentum they are $L_m^\dagger(\hat{p}^a) = L_{-m}(\hat{p}^a)$.

In either case we find, when we normal-order the modified Virasoro operators (4.1a) using the string-type ordering, that the center depends upon the background charges Q^a ,

$$c = 2 - 12Q_a Q^a. \quad (4.3)$$

In BRST quantization, we add ghost fields c_m, \bar{c}_m and antighost fields b_m, \bar{b}_m to the theory, which contribute $c_{\text{gh}} = -26$ to the center. The BRST charge is then found to

be nilpotent only when the total center vanishes, so we fix Q^a by setting

$$c_{\text{TOT}} = -24 - 12Q_a Q^a = 0. \quad (4.4)$$

The BRST cohomology for this theory has been computed [9]. It has a continuous component, consisting of two families of states, labeled by the zero-mode momenta of the fields,

$$|\psi\rangle_{\text{BRST}} = |p^\pm\rangle_{\text{ST}} \otimes c_1 \bar{c}_1 |0\rangle_{\text{gh}}. \quad (4.5)$$

[The state $|0\rangle_{\text{gh}}$ is the $SL(2, \mathbf{C})$ ghost vacuum. The states $|\psi\rangle_{\text{BRST}}$, which are annihilated by the positive-frequency ghost and antighost oscillators, are defined to have ghost number zero.] These states correspond to the solutions (3.6) that we found in the previous Section, using the ordering appropriate to the functional Schrödinger representation.

There is, in addition, a “discrete” component in the BRST spectrum that has its counterpart in the two states (3.7). These additional states appear at ghost number 0, ± 1 when the zero-mode momentum eigenvalues p^\pm satisfy

$$p^+ = irQ^+ \quad (4.6a)$$

$$p^- = isQ^-, \quad (4.6b)$$

where r and s are integers, with $rs > 0$. When the background charge is nonzero, this gives an infinite tower of states labeled by imaginary eigenvalues of the zero-mode momentum.

If the operator \hat{p}^a is anti-Hermitian, then its eigenvalues are imaginary, and the discrete cohomology states are in the spectrum of the theory. On the other hand, if \hat{p}^a is Hermitian, then we exclude these states from the spectrum, which then contains only the continuous family of states (4.5). In the functional Schrödinger quantization

performed in the previous Section we found two discrete states at vanishing zero-mode momentum. Since we did not need to introduce a background charge in that approach, those states in a sense correspond to the case $Q^\pm \rightarrow 0$ in (4.6a,b). However, in the BRST quantization, the case $r = s = 0$ does not lead to a new state; consequently, the BRST spectrum differs from the functional Schrödinger spectrum both for Hermitian and anti-Hermitian \hat{p}^a .

5 Coupling to Scalar Matter

In Sections 1-4 we studied the quantization of the pure dilaton-gravity theory whose action is given in Eq. (1.1). We now introduce matter in the form of a massless scalar field f minimally coupled to the metric tensor field,

$$S = \int dt \int d\sigma \sqrt{-g} \left[\frac{1}{4\pi G} (\eta R - 2\lambda) + \frac{1}{2} g^{\mu\nu} \partial_\mu f \partial_\nu f \right]. \quad (5.1)$$

Here G is the gravitational coupling constant (“Newton’s constant”), and we continue to take space-time to be a cylinder, so that σ runs from 0 to 2π . (The generalization to N matter fields is straightforward but unnecessary; we do not consider the large- N limit.) As stated in the Introduction, it is known that this theory cannot be quantized without modification, due to an obstruction [7]. In subSection 5.1 we modify the action by a term that has been considered previously [7, 12], and present a quantization of the modified theory in the functional Schrödinger formalism, following the treatment in [12]. In subSection 5.2 we follow the development in [11, 14] to perform a BRST quantization of the theory, using a string-like ordering for the gravitational fields, and compare the different results of the two methods.

It is clear that (5.1) may be reduced to a sum of three free scalar field actions, with indefinite kinetic terms. The reduction for the gravity portion proceeds as previously

described (with obvious insertions of the factor $4\pi G$) while the matter already is in that form. In terms of the redefined gravitational fields, (5.1) in first order form becomes

$$S = \int dt \int d\sigma \left(\pi_a \dot{r}^a + \Pi \dot{f} - \mathcal{H} \right), \quad (5.2)$$

where the matter variables Π and f are canonically conjugate. The Hamiltonian density is $\mathcal{H} = u\mathcal{E} + v\mathcal{P}$, with

$$\mathcal{E} = -\frac{1}{2} \left(G\pi_a\pi^a + \frac{1}{G}r'_a r'^a \right) + \frac{1}{2}(\Pi^2 + f'^2) \quad (5.3)$$

$$\mathcal{P} = -\pi_a r'^a - \Pi f'. \quad (5.4)$$

(As before, λ has been set to 4π .) It is clear from these expressions that there is a center in the constraint algebra. As discussed in Section 1, there is a contribution to the center from each of the fields r^0 , r^1 , and f . Those arising from r^0 and r^1 can either add or cancel, depending upon the ordering that is chosen. In either case, there is a nonvanishing center from the f field, and this remains true if we increase the matter content, since each of the matter fields will contribute to the center with the same sign.

It is amusing to note that the above argument can be circumvented if we adopt peculiar ordering conventions for each of the three fields [18], and in the Appendix we show that it is possible to vary the center continuously by ordering in this fashion with respect to a class of generalized “squeezed” vacuum states. However, our construction does not lead to a well-defined quantum theory because of divergences that continue to plague the formalism [19].

5.1 Schrödinger quantization of the modified theory

In this subSection we demonstrate that, by adding terms dependent upon the gravitational fields to the matter-coupled action (5.2), we can construct consistent quantum theories using the functional Schrödinger formalism [7, 12]. The additional terms

canonically introduce a center in the Poisson bracket algebra of the constraints that cancels the quantum anomaly. Their coefficients are proportional to \hbar (set to unity in all of our expressions); in the classical limit, the additional terms vanish, so the modified action corresponds in this limit to the unmodified one (5.2).

As a starting point, we use the first-order form (5.2) for the matter-gravity action, with the constraints given in (5.3), (5.4). We change variables in the gravitational sector of the theory, from π_a, r^a to new fields P_\pm, X^\pm ; in the new variables we can make rapid contact with the results of [12], which we shall use below to quantize the modified theory.

Up to this point it has not been necessary to specify the signs of G and λ ; they occur in the combination $4\pi G/\lambda$, and without loss of generality we had set $\lambda = 4\pi$. (The results for pure dilaton-gravity are insensitive to the sign of λ .) However, below we shall argue that in order to use the results of [12], $4\pi G/\lambda$ should be negative, *i.e.* we take $G = -|G|$. The reasons for making this choice become apparent later in the discussion.

The new fields are defined by the following transformation [7, 12],

$$P_\pm = -\frac{\sqrt{|G|}}{2}(\pi_0 + \pi_1) \mp \frac{1}{2\sqrt{|G|}}(r^{0'} - r^{1'}) , \quad (5.5a)$$

$$X^{\pm'} = \mp \frac{\sqrt{|G|}}{2}(\pi_0 - \pi_1) - \frac{1}{2\sqrt{|G|}}(r^{0'} + r^{1'}) . \quad (5.5b)$$

When we write (5.2) in terms of P_\pm and X^\pm , it is convenient to express the action in terms of constraints C_\pm , which are light-cone combinations of the constraints \mathcal{E}, \mathcal{P} defined in (5.3) and (5.4);

$$C_\pm = -\frac{1}{2}(\mathcal{P} \mp \mathcal{E}) = P_\pm X^{\pm'} \pm \frac{1}{4}(\Pi \pm f')^2 . \quad (5.6)$$

With these definitions, (5.2) appears as

$$S = \int dt \int d\sigma \left[P_+ \dot{X}^+ + P_- \dot{X}^- + \Pi \dot{f} - \mu_+ C_+ - \mu_- C_- \right] , \quad (5.7)$$

where

$$\mu_{\pm} = \pm u - v . \quad (5.8)$$

The constraints C_{\pm} satisfy a commutator algebra that is determined by (5.6), (1.15a), and (1.17), trivially modified to a quantization on a circle, rather than on an infinite line.

$$i[C_{\pm}(\sigma), C_{\pm}(\tilde{\sigma})] = -\left(C_{\pm}(\sigma) + C_{\pm}(\tilde{\sigma}) \pm \frac{1}{24\pi}\right) \delta'(\sigma - \tilde{\sigma}) \pm \frac{1}{24\pi} \delta'''(\sigma - \tilde{\sigma}) , \quad (5.9a)$$

$$i[C_+(\sigma), C_-(\tilde{\sigma})] = 0 . \quad (5.9b)$$

There is, as before, a center $c = 1$, coming solely from the matter fields, and an additional contribution $\mp(1/24\pi)\delta'(\sigma - \tilde{\sigma})$ because we are quantizing on a circle, which was absent in Eq. (1.17) of Section 1, where we quantized on a line; unlike the triple derivative term, the single derivative addition is trivial: it can be removed by adding $\mp 1/48\pi$ to the constraint operators. The transformation (5.5a,b) to X^{\pm} , P_{\pm} , and use of the constraints (5.6), relies on the fact that the pure gravity portion of the constraints is anomaly free: $P_{\pm} X^{\pm'}$ is like a field theoretic momentum density, which has no anomalies in its algebra.

We now introduce the modified action \tilde{S} ,

$$\tilde{S} = \int dt \int d\sigma \left[P_+ \dot{X}^+ + P_- \dot{X}^- + \Pi \dot{f} - \mu_+ \tilde{C}_+ - \mu_- \tilde{C}_- \right] , \quad (5.10)$$

where the new constraints \tilde{C}_{\pm} are obtained from C_{\pm} by removing the trivial modification, and by adding a term proportional to the center and dependent upon the

gravitational “coordinate” fields X^\pm [7, 12],[¶]

$$\tilde{C}_\pm \equiv C_\pm \pm \frac{1}{48\pi} [\ln(\pm X^{\pm'})]'' \mp \frac{1}{48\pi} . \quad (5.11)$$

The constraints \tilde{C}_\pm then satisfy an algebra without center,

$$i[\tilde{C}_\pm(\sigma), \tilde{C}_\pm(\tilde{\sigma})] = -\left(\tilde{C}_\pm(\sigma) + \tilde{C}_\pm(\tilde{\sigma})\right)\delta'(\sigma - \tilde{\sigma}) , \quad (5.12)$$

and, as a consequence, the Dirac quantization conditions $\tilde{C}_\pm|\psi\rangle = 0$ are consistent. However, we are now faced with the problem of solving the constraint equations, which include a complicated, non-polynomial term, dependent upon the gravitational field variables. Nevertheless, a solution to similar constraint equations was reported in [12]; we adopt that argument for our purposes below. ^{||}

The crucial observation of [12] is that the modified action (5.10) is canonically equivalent to one in which the constraint conditions have a simple form. This canonical transformation relies on an expansion of the matter fields in terms of “gravitationally dressed” mode operators a_n^\pm ,

$$a_n^\pm \equiv \frac{1}{2\sqrt{\pi}} \int_0^{2\pi} d\sigma e^{inX^\pm} (\Pi \pm f') . \quad (5.13)$$

(With this normalization, the dressed operators satisfy the usual commutator algebra $[a_m^\pm, a_n^\pm] = m\delta_{m+n,0}$, $[a_m^+, a_n^-] = 0$.)

Eq. (5.13) can be inverted, and expressions for Π and f' obtained, when the fields X^\pm satisfy

$$X^\pm(2\pi) - X^\pm(0) = \pm 2\pi , \quad (5.14a)$$

and

$$X^{\pm'} \neq 0 \quad \text{everywhere.} \quad (5.14b)$$

[¶]This is equivalent to adding a background charge: see Ref. [7].

^{||}E.B. is grateful to T. Strobl for enlightening discussions about the work in [12].

Together, these equations require that X^+ be monotonically increasing, and that X^- be monotonically decreasing. We show that this is consistent with the combined transformations (1.7a-d) and (5.5a,b). For $X^{+'}$ we find

$$X^{+'} = |G|^{-1/2} \exp(\rho - 4\pi|G|\Sigma) > 0, \quad (5.15)$$

which is clearly consistent with (5.14a,b). The expression for $X^{-'}$ does not have such a simple form; however, P_- does:

$$P_- = -|G|^{-1/2} \exp(\rho + 4\pi|G|\Sigma), \quad (5.16)$$

while from the expression (5.6) for the unmodified constraints we find that $C_- = 0$ implies

$$P_- X^{-'} = \frac{1}{4}(\Pi - f')^2, \quad (5.17)$$

so that on the $C_- = 0$ surface,

$$X^{-'} = -\frac{\sqrt{|G|}}{4}(\Pi - f')^2 \exp(\rho + 4\pi|G|\Sigma) < 0. \quad (5.18)$$

It is here that we see that we need to take G negative. Had we taken G to be positive, we would have found $P_- > 0$, which would then have implied that X^- is monotonically increasing, rather than decreasing, when $C_- = 0$. [The condition that the modified constraints vanish, $\tilde{C}_- = 0$, does not yield such a straightforward result. However, we note that the added term $-(1/48\pi)[\ln(\pm X^{-'})]''$ in (5.11) is singular at $X^{-'} = 0$, so $X^{-'}$ should be strictly positive or strictly negative, and it is reasonable that it should be strictly negative, in agreement with the classical result (5.18).]

Inverting (5.13) we find

$$\Pi = \frac{1}{2\sqrt{\pi}} \left(X^{+'} \sum_n e^{-inX^+} a_n^+ - X^{-'} \sum_n e^{-inX^-} a_n^- \right) \quad (5.19)$$

$$\begin{aligned}
f' &= \frac{1}{2\sqrt{\pi}} \left(X^{+'} \sum_n e^{-inX^+} a_n^+ + X^{-'} \sum_n e^{-inX^-} a_n^- \right) \\
f &= \frac{1}{2\sqrt{\pi}} \left(i \sum_{n \neq 0} \frac{1}{n} e^{-inX^+} a_n^+ + i \sum_{n \neq 0} \frac{1}{n} e^{-inX^-} a_n^- + X^+ a_0^+ + X^- a_0^- + \hat{x} \right), \quad (5.20)
\end{aligned}$$

where \hat{x} is independent of σ . Because f is periodic, a_0^\pm are identified; we rename them \hat{p} since they are conjugate to \hat{x} ,

$$\hat{p} \equiv a_0^+ = a_0^- . \quad (5.21)$$

We now reorder the matter fields. This is convenient for further development, and does not affect the center. We order the fields with respect to an eigenstate of \hat{p} that is annihilated by the positive frequency gravitationally dressed mode operators,

$$\hat{p}|p\rangle = p|p\rangle \quad (5.22a)$$

$$a_n^\pm |p\rangle = 0, \quad n > 0. \quad (5.22b)$$

This changes the constraint algebra to

$$i[\tilde{C}_\pm(\sigma), \tilde{C}_\pm(\tilde{\sigma})] = -\left(\tilde{C}_\pm(\sigma) + F_\pm(\sigma) + \tilde{C}_\pm(\tilde{\sigma}) + F_\pm(\tilde{\sigma}) \right) \delta'(\sigma - \tilde{\sigma}) \quad (5.23a)$$

$$F_\pm = \pm(1/48\pi) \left[4\sqrt{\pm X^{\pm'}} \left(\frac{1}{\sqrt{\pm X^{\pm'}}} \right)'' + (X^{\pm'})^2 \right], \quad (5.23b)$$

and we must introduce further terms in the modified constraints (5.11) to cancel the new ordering-dependent contributions F_\pm . The final constraints then read

$$\overline{C}_\pm = C_\pm \mp \frac{1}{48\pi} X^{\pm'} \left[X^{\pm'} + \left(\frac{1}{X^{\pm'}} \right)'' \right], \quad (5.24)$$

and they satisfy the algebra (5.12), without center.

The dressed mode operators (5.13) all commute with \overline{C}_\pm in (5.24), and also with X^\pm . With this information it is straightforward to verify that the following transformation [12],

$$\overline{X}^{\pm'} \overline{P}_\pm = \overline{C}_\pm \quad (5.25a)$$

$$\overline{X}^\pm = X^\pm \quad (5.25b)$$

$$\frac{1}{2\sqrt{\pi}} \left[\overline{X}^{+'} \sum_{n \neq 0} e^{-in\overline{X}^+} a_n^+ - \overline{X}^{-'} \sum_{n \neq 0} e^{-in\overline{X}^-} a_n^- + (\overline{X}^+ - \overline{X}^-)' \hat{p} \right] = \Pi \quad (5.25c)$$

$$\frac{1}{2\sqrt{\pi}} \left[i \sum_{n \neq 0} \frac{1}{n} e^{-in\overline{X}^+} a_n^+ + i \sum_{n \neq 0} \frac{1}{n} e^{-in\overline{X}^-} a_n^- + (\overline{X}^+ + \overline{X}^-) \hat{p} + \hat{x} \right] = f, \quad (5.25d)$$

is canonical at the quantum level.

In terms of the new fields \overline{X}^\pm , \overline{P}_\pm , the constraints take the simple form $\overline{C}_\pm = \overline{X}^{\pm'} \overline{P}_\pm$, and we can now quantize the theory in these variables following the Dirac procedure. Allowing the momenta \overline{P}_\pm to act by functional differentiation, and \overline{X}^\pm by multiplication, we see that the condition that \overline{C}_\pm annihilate physical states requires that the states are independent of \overline{X}^\pm . The spectrum therefore consists of $|p\rangle$, defined in (5.22a,b), and states constructed by acting on it with the negative-frequency dressed operators $a_{-|n|}^\pm$, subject to a further condition [12], as we now explain.

The operators $a_{-|n|}^+$ and $a_{-|n|}^-$ cannot be applied independently of one another, because of a constraint arising from the periodic boundary conditions for the fields r^a : from the transformation (5.5a,b) that defines X^\pm , P_\pm , we find that $P_+ - P_- = \sqrt{\Lambda} (r^{0r} - r^{1r})$, so the integral over the circle of $P_+ - P_-$ vanishes,

$$\int_0^{2\pi} d\sigma (P_+ - P_-) = 0. \quad (5.26)$$

Evaluating

$$\int_0^{2\pi} \left(\frac{1}{\overline{X}^{+'}} \overline{C}_+ - \frac{1}{\overline{X}^{-'}} \overline{C}_- \right) = \int_0^{2\pi} d\sigma (P_+ - P_-) + \frac{1}{2} \sum_{n \neq 0} (: a_n^+ a_{-n}^+ : - : a_n^- a_{-n}^- :), \quad (5.27)$$

we find that physical states $|\psi\rangle$ must satisfy $\sum_{n \neq 0} (: a_n^+ a_{-n}^+ : - : a_n^- a_{-n}^- :)|\psi\rangle = 0$, which relates the “+” and “−” oscillators appearing in the state. This is the “level-matching condition” familiar from string theory, which takes the above simple form only after the reordering effected above—that is the reason for reordering.

The solutions thus obtained are expressed in terms of the gravitationally dressed mode operators. For some purposes this representation is sufficient: in particular, in the next subSection, we shall be able to compare directly the spectrum found above with that obtained in BRST quantization. However, to make contact with the original geometrical theory, it is desirable to represent the solutions in terms of the variables P_{\pm} , X^{\pm} , Π , and f that enter the action (5.7). We now show that this cannot be done without first solving the constraints (5.24) in terms of those variables.

We find it convenient for this calculation to write the dressed mode operators a_n^{\pm} in terms of harmonic oscillator coordinates and momenta Φ_n^{\pm} and Π_n^{\pm} satisfying $i[\Pi_m^{\pm}, \Phi_n^{\pm}] = \delta_{mn}$,

$$a_n^{\pm} \equiv \frac{1}{\sqrt{2}} \left(\Pi_{|n|}^{\pm} - in \Phi_{|n|}^{\pm} \right). \quad (5.28)$$

We now compute the overlap matrix $\langle f X^{\pm} | \Phi_n^{\pm} \bar{X}^{\pm} \rangle$ that relates the two sets of canonical fields. We obtain a set of functional differential equations for the matrix elements from the equations defining the transformation, (5.25a-d), by promoting the fields to operators, and evaluating matrix elements of the transformation equations. The solution is presented in terms of a functional \mathcal{M} ,

$$\langle f X^{\pm} | \Phi_n^{\pm} \bar{X}^{\pm} \rangle = \delta(\bar{X}^{+} - X^{+}) \delta(\bar{X}^{-} - X^{-}) e^{i \int d\sigma [h^{+} h^{-'} - f'(h^{+} - h^{-})]} \mathcal{M}, \quad (5.29)$$

where

$$h^{\pm} = \sqrt{\frac{2}{\pi}} \left(\frac{1}{2} \hat{x} + \sum_{n=1}^{\infty} \Phi_n^{\pm} \cos nX^{\pm} \right), \quad (5.30)$$

and $\mathcal{M} = \mathcal{M}(X^{+}, X^{-}, f - h^{+} - h^{-})$ satisfies

$$\left\{ X^{\pm'} \frac{1}{i} \frac{\delta}{\delta X^{\pm}} \pm \frac{1}{4} \left(\frac{1}{i} \frac{\delta}{\delta f} \pm f' \right)^2 \mp \frac{1}{48\pi} X^{\pm'} \left[X^{\pm'} + \left(\frac{1}{X^{\pm'}} \right)'' \right] \right\} \mathcal{M}(X^{+}, X^{-}, f) = 0. \quad (5.31)$$

In order to determine the matrix elements we must solve the equations for \mathcal{M} , but they

are seen to be identical to the condition that the constraints $\overline{\mathcal{C}}_{\pm}$ annihilate physical states.

5.2 BRST quantization with background charges

In this subSection we present a BRST quantization of the matter-coupled theory. We adopt a string-like ordering for the fields, so that the total center from the gravitational and matter fields is $c = 3$. We introduce ghosts, which decrease the total center, and background charges, which we adjust so that the center vanishes. Following the development in [11, 14], which relies upon the work of [9], we find a space isomorphic to the ghost-number zero states in the cohomology of the BRST charge. However, our analysis differs from that of [11, 14]; as with the pure gravity theory, there are states in the cohomology with imaginary eigenvalues of the zero-mode momenta. They do not appear in our case, since we take the fields to be Hermitian. The spectrum then obtained has one more zero-mode degree of freedom than that found in the functional Schrödinger approach; otherwise, the spectra coincide. (The fact that there are more zero-mode degrees of freedom in the BRST states than the one associated with the matter field was pointed out and discussed in [7].)

Starting with the action (5.2) we work in conformal gauge, $u = 1$, $v = 0$. To facilitate comparison with [11, 14] we work with rescaled fields, $r^a \rightarrow \sqrt{4\pi/|G|}r^a$, $f \rightarrow \sqrt{4\pi}f$. (The sign of G does not affect the calculations to be performed below; however, to be consistent with the calculation in the previous subSection we take G to be negative, $G = -|G|$.) As in Section 2 we expand the fields in modes

$$r^a(t, \sigma) = \hat{x}^a + 2t\hat{p}^a + i \sum_{n \neq 0} \frac{1}{n} \left[\alpha_n^a e^{-in(t-\sigma)} + \overline{\alpha}_n^a e^{-in(t+\sigma)} \right], \quad (5.32a)$$

$$f(t, \sigma) = \hat{x}^M + 2t\hat{p}^M + i \sum_{n \neq 0} \frac{1}{n} \left[\alpha_n^M e^{-in(t-\sigma)} + \overline{\alpha}_n^M e^{-in(t+\sigma)} \right]. \quad (5.32b)$$

We note that because of the choice of sign for G , now r^0 has positive kinetic term, while that of r^1 is negative. Consequently, the commutators of the mode operators differ by an interchange $0 \leftrightarrow 1$ from those of Section 2. They are now

$$[\hat{p}^a, \hat{x}^b] = -i\eta^{ab} \quad (5.33)$$

$$[\alpha_m^a, \alpha_n^b] = [\bar{\alpha}_m^a, \bar{\alpha}_n^b] = m\eta^{ab}\delta_{m+n,0}. \quad (5.34)$$

We order with respect to an eigenstate $|p^a, p^M\rangle$ of the zero-mode momentum operators that is annihilated by the positive frequency mode operators,

$$\alpha_m^a |p^a, p^M\rangle = \alpha_m^M |p^a, p^M\rangle = 0, \quad m > 0 \quad (5.35)$$

$$\hat{p}^a |p^a, p^M\rangle = p^a |p^a, p^M\rangle, \quad (5.36)$$

$$\hat{p}^M |p^a, p^M\rangle = p^M |p^a, p^M\rangle. \quad (5.37)$$

(We shall only record explicit expressions for the unbarred degrees of freedom.) The Virasoro operators, modified by the addition of background charges Q^a to the gravitational fields, take the following form,

$$L_m = \frac{1}{2} \sum_{n \neq -m, 0} (\eta_{ab} : \alpha_{m+n}^a \alpha_{-n}^b : + : \alpha_{m+n}^M \alpha_{-n}^M :) \\ + \hat{p}_a \alpha_m^a + \hat{p}^M \alpha_m^M + im Q_a \alpha_m^a, \quad m \neq 0, \quad (5.38a)$$

$$L_0 = \frac{1}{2} \sum_{n \neq 0} (\eta_{ab} : \alpha_n^a \alpha_{-n}^b : + : \alpha_n^M \alpha_{-n}^M :) + \frac{1}{2} \hat{p}_a \hat{p}^a + \frac{1}{2} (\hat{p}^M)^2 - \frac{1}{2} Q^a Q_a. \quad (5.38b)$$

As in Section 4, we take the momenta to be Hermitian and the background charges to be real, to be consistent with the functional Schrödinger quantization of the previous subSection. With the contribution from the background charges, the center of the Virasoro algebra is given by

$$c = 3 + 12Q_a Q^a. \quad (5.39)$$

In BRST quantization, we add ghost fields c_m , \bar{c}_m and antighost fields b_m , \bar{b}_m which contribute $c_{gh} = -26$ to the center. Nilpotency of the BRST charge is accomplished only when the total center vanishes,

$$c_{\text{TOT}} = c + c_{\text{gh}} = -23 + 12Q_a Q^a = 0. \quad (5.40)$$

We use this condition to fix the value of $Q_a Q^a$.

Our task is now to find the ghost-number zero states in the cohomology of the BRST charge $d_{\text{BRST}} \equiv d + \bar{d}$, which we identify as “physical”. The unbarred fields contribute to d_{BRST} the quantity d ,

$$d = \sum_{n=-\infty}^{\infty} c_{-n} L_n - \frac{1}{2} \sum_{m,n=-\infty}^{\infty} : (m-n) c_{-m} c_{-n} b_{m+n} :. \quad (5.41)$$

(The contribution \bar{d} from the barred fields has an identical form.) Since the barred and unbarred operators act independently (aside from the level-matching condition, discussed at the end of this subSection), it is sufficient to compute the cohomology of d , rather than d_{BRST} .

In [9], a strategy for finding the d -cohomology was proposed. It was shown that it is sufficient to compute the cohomology of a much simpler operator. The appropriate operator emerges after several steps. First, we define \hat{d} as all of the terms in d that do not contain the zero-mode ghost operators c_0 and b_0 . We find that d can be written

$$d = c_0(L_0 + L_0^{gh}) + b_0 M + \hat{d}, \quad (5.42)$$

where M contains only ghost fields, L_0 is defined in (5.38b), and L_0^{gh} is the ghost Virasoro operator $L_0^{gh} = \sum_m m : c_{-m} b_m :$. From (5.42) we see that $L_0 + L_0^{gh} = \{b_0, d\}$. A standard argument** shows that a nontrivial element of the cohomology must be

** Suppose that $|\psi\rangle$ is an element of the d -cohomology and $(L_0 + L_0^{gh})|\psi\rangle = h|\psi\rangle$, $h \neq 0$. Then we can write $|\psi\rangle$ as $|\psi\rangle = d(b_0/h|\psi\rangle)$, which is a trivial (d -exact) state.

annihilated by

$$L_0 + L_0^{gh} \equiv p^+ p^- + \frac{1}{2}(p^M)^2 - Q^+ Q^- + \hat{L} - 1 = 0, \quad (5.43)$$

where \hat{L} is the total “level operator” including gravity, scalar matter and ghosts. Next, we introduce the subspace \mathcal{F} of states annihilated by $L_0 + L_0^{gh}$ and b_0 ,

$$\mathcal{F} = \{|\psi\rangle | (L_0 + L_0^{gh})|\psi\rangle = 0 \text{ and } b_0|\psi\rangle = 0\}. \quad (5.44)$$

From (5.42) we see that an element $|\psi\rangle$ of \mathcal{F} annihilated by \hat{d} is also annihilated by d ; furthermore, \hat{d} is nilpotent on \mathcal{F} , and we can consistently compute its cohomology in that space. It is possible to show [9] that the d -cohomology can be constructed from the \hat{d} -cohomology defined on \mathcal{F} : to each $|\psi\rangle$ in the \hat{d} -cohomology there correspond two possible elements, $|\psi\rangle$ and $c_0|\psi\rangle$, in the d -cohomology. However, we are interested in states with no ghost excitations, and they are given by the \hat{d} -cohomology.

We now introduce the operator whose cohomology is isomorphic to that of \hat{d} . Following [9], we assign “degrees” to the mode operators

$$\deg(\alpha_n^+) = \deg(c_n) = 1, \quad (5.45a)$$

$$\deg(\alpha_n^-) = \deg(b_n) = -1, \quad (n \neq 0). \quad (5.45b)$$

The degrees of all other operators are defined to be zero. Then \hat{d} is a sum of terms of degree 0, 1 and 2, $\hat{d} = \hat{d}_0 + \hat{d}_1 + \hat{d}_2$. The contribution to \hat{d} of zero degree is

$$\hat{d}_0 = \sum_{n \neq 0} P^+(n) c_{-n} \alpha_n^-, \quad (5.46)$$

where

$$P^+(n) = p^+ + iQ^+ n. \quad (5.47)$$

It was shown in [9] that there is a one to one correspondence between the \hat{d}_0 and \hat{d} cohomologies (and also those of \hat{d}_2 and \hat{d}), so it is sufficient for our purpose to compute the \hat{d}_0 -cohomology, which is a relatively simple problem.

We use a trick described in [9] to find the states. We define the operator

$$K \equiv \sum_{n \neq 0} \frac{1}{P^+(n)} \alpha_{-n}^+ b_n, \quad (5.48)$$

which satisfies

$$\{\hat{d}_0, K\} = \sum_{n \neq 0} (n c_{-n} b_n + \alpha_{-n}^+ \alpha_n^-) \equiv \hat{L}_{gg}, \quad (5.49)$$

where \hat{L}_{gg} is the contribution to the level operator \hat{L} from the gravity and ghost fields. (Note that K is well defined for all real values of p^+ since $P^+(n)$ never vanishes.)

Eq. (5.49) implies that a state in the \hat{d}_0 -cohomology is annihilated by \hat{L}_{gg} , using the same argument that led us to conclude that $L_0 + L_0^{gh}$ [Eq. (5.43)] annihilates nontrivial states in the d -cohomology; consequently the nontrivial \hat{d}_0 -cohomology states have neither gravity nor ghost excitations. Thus the \hat{d}_0 -cohomology is the set of all states constructed by acting an arbitrary number of times with the negative-frequency matter oscillators $\alpha_{-|n|}^M$ on the vacuum state $|p^\pm, p^M\rangle$. In addition, if the states are to lie in the subspace \mathcal{F} , as we assumed, then they must also satisfy the condition that they are annihilated by $L_0 + L_0^{gh}$, Eq. (5.43). (We note that from a theorem of [9] we can, if we wish, construct the explicit \hat{d} -cohomology states from those of the \hat{d}_0 operator: the procedure is explained in [9] and [14].)

We have constructed the ghost-number zero states in the cohomology of the unbarred operator d . In order to obtain the cohomology of the full BRST charge d_{BRST} we must consider the \bar{d} -cohomology as well. As previously mentioned, the \bar{d} -cohomology is just a copy of that of the d operator. In the d_{BRST} -cohomology there is, however, an additional condition found by applying $[(L_0 + L_0^{gh}) - (\bar{L}_0 + \bar{L}_0^{gh})]$ to physical states $|\psi\rangle$. This is the level-matching condition,

$$\sum_n (: \alpha_n^M \alpha_{-n}^M : - : \bar{\alpha}_n^M \bar{\alpha}_{-n}^M :) |\psi\rangle = 0. \quad (5.50)$$

The physical states in the d_{BRST} -cohomology are thus obtained by applying the $\alpha_{-|n|}^M$ and $\bar{\alpha}_{-|n|}^M$ oscillators to $|p^\pm, p^M\rangle$, subject to the condition (5.50).

At this point we note that had we allowed states with imaginary momenta as in [11, 14], we would have found a larger spectrum. In that case, the operator K is not always well defined since there exist momenta p^+ , and non-zero integers n , such that $P^+(n) = 0$. In the construction above, states with these momenta must be treated as special cases; these are the discrete states. However, we work with Hermitian fields, so we exclude the discrete states from the spectrum.

We now compare the spectrum that we have obtained with that described in Subsection 5.1 using the functional Schrödinger formalism. The latter states were obtained by acting independently with an arbitrary number of the negative-frequency dressed operators $a_{-|n|}^+$ and $a_{-|n|}^-$ of the scalar matter field on the vacuum state, subject to the level matching condition. The spectrum is therefore very similar to the one presented above. The BRST spectrum is also constructed by applying two sets of negative-frequency creation operators, $\alpha_{-|m|}^M$ and $\bar{\alpha}_{-|m|}^M$, again subject to the level-matching condition. However, there is a difference between the two spectra [7]. The states obtained with the functional Schrödinger formalism are labeled only by the zero-mode momentum of the matter field, p^M . In BRST quantization, we have three zero-modes (p^+, p^-, p^M) and one constraint, Eq. (5.43), that restricts the momenta. The BRST states are thus labeled by two parameters, which was also pointed out in [7].

6 Conclusion and Discussion

In this paper we discussed two different approaches to quantizing the string-inspired model of two dimensional gravity, the functional Schrödinger and BRST methods. We treated both the pure dilaton-gravity theory, and dilaton-gravity coupled to scalar

matter. Our main tool in this task was a sequence of field redefinitions [7, 12], which we used to express the constraint conditions in alternative forms, in which we recognized problems that have been discussed in the literature [7, 9, 10, 11, 12]. For both the matter-coupled and pure gravity theories we found different spectra using the two different quantization procedures.

In the case of pure gravity, there are two states in the functional Schrödinger spectrum that have no counterpart in the BRST approach (although in a sense they correspond to the discrete states that appear at imaginary momentum in the BRST cohomology); otherwise the spectra coincide.

With the matter-coupled theory, there is a discrepancy in the zero-mode degrees of freedom in the two spectra. The extra, gravitational, zero-mode that appears in BRST quantization was identified and commented upon in [7]. Its presence was considered problematic, since the spectrum does not then resemble that of a massless scalar field on a flat space-time, as one would expect from the classical analysis. The functional Schrödinger spectrum, on the other hand, obtained using the approach of [12], does not share that difficulty, since there are no free zero-modes in the gravitational degrees of freedom. Consequently, for the matter-coupled theory the functional Schrödinger approach yields the most “natural” spectrum.

When we quantized the matter-coupled theory we argued that the requirement that X^+ and X^- be monotonically increasing and decreasing, respectively [Eqs. (5.14a,b)], could only be met when $4\pi G/\lambda$ is negative. As we emphasized in the body of the paper, the sign of $4\pi G/\lambda$ has no effect on any other calculation that we performed; however, it does make a difference in the CGHS model [1], from which our action (1.1) is derived [4, 7], since black hole solutions in the CGHS metric exist only for positive cosmological constant. On the other hand, we do not expect that the restriction we encountered

is generic, for the following reason. The constraint on the sign arose from considering the field redefinitions (1.7a-d) and (5.5a,b), and the form of those transformations, in turn, is strongly constrained by the requirement that the fields r^a be periodic on the circle. The quantization that we have performed on the circle should carry over (with obvious modifications) to quantization on a finite interval, where more general boundary conditions can be applied, and for which the restriction on the sign of $4\pi G/\lambda$ need not hold. (A nice discussion of boundary conditions is given in Ref. [20].)

Having obtained the physical states in the matter-coupled theory, a natural next step is to try to extract the space-time geometry arising from a particular distribution of matter fields. However, at present it is not possible to do this, because we cannot explicitly express the solutions in terms of the dilaton η and the field ρ that we used to parameterize the metric tensor.

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Appendix

We show that we can order the constraints (5.3), (5.4) so that their commutators satisfy an algebra without center. We order the constraint operators with respect to a class of vacuum states annihilated by linear combinations of positive and negative frequency mode operators; that is, by linear combinations of the usual “annihilation”

and “creation” operators.^{††} Such states are studied in quantum optics, and are called “squeezed states”. We shall find, however, that states built upon these squeezed vacua are not invariant under finite action of the constraints, due to divergences that arise when we order products of the constraint operators.

As in Section 2, we expand the fields r^a and f entering (5.1), and the constraints (5.3), (5.4) in terms of mode operators. We work in conformal gauge, which we fix by setting $u = 1$ and $v = 0$ in the parameterized metric tensor (1.2). We work with the rescaled fields of subSection 5.2, $r^a \rightarrow \sqrt{4\pi/|G|}r^a$, $f \rightarrow \sqrt{4\pi}f$, which we expand in terms of mode operators as in Eqs. (5.32a,b). (As discussed previously, taking $G = -|G|$ to be negative implies that the “0” field has positive kinetic term, while the “1” field has negative kinetic term.) The Virasoro operators have the form

$$L_m = \frac{1}{2} \sum_{n \neq -m, 0} (\eta_{ab} : \alpha_{m+n}^a \alpha_{-n}^b : + : \alpha_{m+n}^M \alpha_{-n}^M :) + \hat{p}_a \alpha_m^a + \hat{p}^M \alpha_m^M, \quad m \neq 0 \quad (\text{A.1a})$$

$$L_0 = \frac{1}{2} \sum_{n \neq 0} (\eta_{ab} : \alpha_n^a \alpha_{-n}^b : + : \alpha_n^M \alpha_{-n}^M :) + \frac{1}{2} \hat{p}_a \hat{p}^a + \frac{1}{2} (\hat{p}^M)^2. \quad (\text{A.1b})$$

(A similar expression holds for the barred operators \bar{L}_m : for the rest of this Appendix, we present explicit expressions only for the unbarred operators.) Both L_m and \bar{L}_m satisfy the Virasoro algebra (2.8) with a center that depends upon the chosen operator ordering.

We now introduce the class of vacuum states that allows us to vary continuously the value for the center. We define mode operators $\tilde{\alpha}_m^a, \tilde{\alpha}_m^M$ by performing the following transformation (a Bogoliubov transformation) for each of the modes $m \neq 0$,

$$\tilde{\alpha}_m^a = \alpha_m^a \cosh \theta_a + \alpha_{-m}^a \sinh \theta_a \quad (\text{A.2a})$$

$$\tilde{\alpha}_m^M = \alpha_m^M \cosh \theta_M + \alpha_{-m}^M \sinh \theta_M \quad (\text{A.2b})$$

^{††}The same approach was used in [18] to construct bosonic string theories in an arbitrary number of target-space dimensions.

where $\{\theta_a, \theta_M\}$ are parameters that we take to be either real or imaginary. We now define the state $|\widetilde{0}\rangle$

$$\tilde{\alpha}_{|n|}^0 |\widetilde{0}\rangle = \tilde{\alpha}_{|n|}^M |\widetilde{0}\rangle = 0 \quad (\text{A.3a})$$

$$\tilde{\alpha}_{-|n|}^1 |\widetilde{0}\rangle = 0 . \quad (\text{A.3b})$$

(Since our conclusions are unaffected by the eigenvalues of \hat{p}^a, \hat{p}^M , we take $|\widetilde{0}\rangle$ to satisfy $\hat{p}^a |\widetilde{0}\rangle = \hat{p}^M |\widetilde{0}\rangle = 0$ for simplicity.) The contribution to the center from each of the fields, when the Virasoro operators are ordered with respect to $|\widetilde{0}\rangle$, is conveniently computed by substituting for α_m^a and α_m^M in terms of $\tilde{\alpha}_m^a$ and $\tilde{\alpha}_m^M$ in (A.1a,b), and then evaluating the commutator

$$[L_m - L_{-m}, L_m + L_{-m}] = 4mL_0 + \frac{c}{6}(m^3 - m) . \quad (\text{A.4})$$

The center is found to depend upon the parameters θ_a, θ_M ,

$$c = \cosh 2\theta_0 - \cosh 2\theta_1 + \cosh 2\theta_M . \quad (\text{A.5})$$

(Note that this gives $c = 1$ when all of the θ 's vanish, as expected.) It is clear that there are many solutions $\{\theta_a, \theta_M\}$ for which $c = 0$.

With the center set to zero it is possible in principle to solve the constraint equations $L_m |\psi\rangle = 0$ for a state $|\psi\rangle$ built upon the vacuum $|\widetilde{0}\rangle$ defined by Eqs. (A.3a,b). There is a difficulty, however, with the interpretation of the solution obtained in this way. In [19], a calculation related to ours suggests that the Virasoro operators ordered with respect to vacua defined by (A.3a,b) cannot be exponentiated to yield finite transformations, because of infinities that appear when products of the constraint operators are ordered with these states.^{‡‡} Below, we show how these infinities arise in our calculation. We

^{‡‡} Moreover, in [18] it was argued that, from the point of view of string theory, the excitations of the string cannot be interpreted as particle states.

shall find that while we can cancel them at quadratic order, new divergences arise at higher orders.

When we evaluate the vacuum expectation value of the square of the Virasoro operator L_m defined in (A.1a), we find

$$\langle \widetilde{0} | L_m^2 | \widetilde{0} \rangle = C \left[\frac{1}{12} |m| (m^2 - 1) + \sum_{n=1}^{\infty} (|m| + n) n \right]. \quad (\text{A.6})$$

where $C = \frac{1}{4}(\sinh^2 2\theta_0 + \sinh^2 2\theta_1 + \sinh^2 2\theta_M)$. The second of the bracketed terms in (A.6) is divergent, and for real values of the parameters each of the fields contributes with the same sign. However, for imaginary θ , $\sinh^2 2\theta$ is negative, and we can arrange a cancelation in the prefactor C by taking some of the parameters to be imaginary, and others to be real. (We also note that the contribution to the center, $\cosh 2\theta$, is real for imaginary θ .) We demonstrate that it is possible to solve simultaneously the conditions of vanishing center and vanishing vacuum expectation value of L_m^2 by recording a particular solution,

$$\theta_0 = \theta_M = \frac{i}{2}\beta, \quad \cos \beta = \frac{1}{\sqrt{2}} \quad (\text{A.7a})$$

$$\cosh 2\theta_1 = \sqrt{2}. \quad (\text{A.7b})$$

However, the cancelation does not persist to higher orders: evaluating $\langle \widetilde{0} | L_m^4 | \widetilde{0} \rangle$ we find

$$\langle \widetilde{0} | L_m^4 | \widetilde{0} \rangle = 3(\langle \widetilde{0} | L_m^2 | \widetilde{0} \rangle)^2 + C_1 \sum_{n=1}^{\infty} (|m| + n)^2 (2|m| + n) n + C_2 \sum_{n=1}^{\infty} (|m| + n)^2 n^2 + \text{finite}, \quad (\text{A.8})$$

where the coefficients C_1 and C_2 are given by $C_i \equiv c_i(\theta_0) + c_i(\theta_1) + c_i(\theta_M)$, with

$$c_1(\theta) = \left[12(\cosh^4 \theta + \sinh^4 \theta) \sinh^2 2\theta + \frac{9}{8} \sinh^4 2\theta \right] \quad (\text{A.9a})$$

$$c_2(\theta) = 3 \sinh^4 2\theta. \quad (\text{A.9b})$$

We see that c_2 is positive even for imaginary θ ; also, c_1 will not vanish in general, when the θ 's are chosen such that (A.6) vanishes. Moreover, it is reasonable to expect that we shall encounter further divergences at higher orders.

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